Math 246B Lecture 4 Notes

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1 Properties of Subharmonic Functions

1.1 Local conditions equivalent to subharmonicity

Last time, we introduced the notion of a subharmonic function.

Theorem 1.1. *Let $u : \Omega \to [-\infty, \infty)$ be upper semicontinuous. The following are equivalent:*

1. *u is subharmonic.*
2. *If $\{|x - a| \leq R\} \subseteq \Omega$, then*
$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} P_R(x - a, y) u(a + y) \, ds(y).$$
3. *(local sub-mean value inequality): For every $a \in \Omega$,*
$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} u(a + y) \, ds(y)$$
for all small $R > 0$.
4. *For every $a \in \Omega$,*
$$u(a) \leq \frac{1}{\pi R^2} \iint_{|y| \leq R} u(a + y) \, dy$$
for all small $R > 0$, where dy is Lebesgue measure in $\mathbb{R}^2$.
5. *If $\{|x - a| \leq R\} \subseteq \Omega$, then*
$$u(a) \leq \frac{1}{\pi R^2} \iint_{|y| \leq R} u(a + y) \, dy$$

Remark 1.1. It follows from properties 3 and 4 that subharmonicity is a local property.

Remark 1.2. The integrals in the theorem are Lebesgue integrals of upper semicontinuous functions. If $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous and $K \subseteq \Omega$ is compact, then

$$\int_K u(x)\, dx = \inf_{\substack{u \le \varphi \\ \varphi \in C(K)}} \int \varphi\, dx \in [-\infty, \infty).$$

Proof. (1) $\implies$ (2): Let $f \in C(|x-a| = R)$, and let $v \in C(|x-a| \le R)$ be harmonic in $|x-a| < R$ so that $v = f$ along $|x-a| = R$. If $u \le f$ on $|x-a| = R$, then $u \le v$ in $|x-a| \le R$. So

$$u(x) \le \frac{1}{2\pi R} \int_{|y|=R} P_R(x-a, y) f(a+y)\, ds(y)$$

for $|x-a| < R$. Pick a sequence $f_k \in C(|x-a| = R)$ such that $f_k \downarrow u$. apply this inequality to every function in the sequence, and let $k \to \infty$ by monotone convergence to get the desired inequality.

(2) $\implies$ (3): Take $x = a$.

(2) $\implies$ (5): If $\{|x-a| \le R\}$, then

$$u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt$$

with $0 < r \le R$. Multiply by $2r$ and integrate over $[0, R]$. This gives us the area integral, expressed in polar coordinates.

(5) $\implies$ (4): This is a special case.

(3) $\implies$ (1): Let $K \subseteq \Omega$ be compact and $h \in C(K) \cap H(K^o)$ such that $u \le h$ on ∂K. We want to show that $u \le h$ on K. The function $u - h$ is upper semicontinuous on K and satisfies the local sub-mean value inequality in K. We can prove the maximum principle for $u - h$ on K with the same proof as for harmonic functions: If $M = \max_K (u-h)$, then the set $\{x \in K : u(x) - h(x) = M\}$ is closed (as $u - h$ is upper semicontinuous on K). We get that $\max_K u - h = \max_{\partial K} \le 0$. So $u \le h$ on K.

(4) $\implies$ (1): The argument is similar to the proof of (3) $\implies$ (1), using the local sub-mean value inequality with respect to small discs rather than circles. $\square$

1.2 Mean value property and maximum principle

In the proof of the theorem, we also proved the following property.

Theorem 1.2 (mean value property for subharmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and bounded, and let u be upper semicontinuous on $\overline{\Omega}$ and subharmonic in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

We also have the following version of the maximum principle.

Theorem 1.3 (maximum principle for subharmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and connected, and let u be subharmonic Ω. If u contains a global maximum on Ω, then it is constant.*

Proof. Let $M = \max_\Omega u$, and notice that the sets $\{u < M\}$, $\{u = M\}$ are open. ☐

It is important to note that the maximum needs to be global. In this sense, subharmonic functions are much less rigid than their harmonic counterparts.

Example 1.1. Here is an example where u attains a local maximum without being constant in Ω. Take $u(z) = \max(0, \mathrm{Re}(z))$.

1.3 Relationship to holomorphic functions

Proposition 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in \mathrm{Hol}(\Omega)$. Then $u = \log|f| : \Omega \to [-\infty, \infty)$ is subharmonic in Ω.*

Proof. We saw before that u is upper semicontinuous, and we shall check that for all $a \in \Omega$,

$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y)$$

for all small $R > 0$. If $f(a) = 0$, then the inequality holds. If $f \neq 0$, then in a small simply connected neighborhood of a, we can write $u = \mathrm{Re}(\log(f))$. Then u is harmonic near a and the inequality holds with an equality for all $R > 0$. ☐

Next time, we will prove the following result.

Proposition 1.2. *Let $f \in C(|z| \leq R) \cap \mathrm{Hol}(|z| < R)$. Assume that there exists a Lebesgue measurable $E \subseteq \{|z| = R\}$ of positive measure such that $f|_E = 0$. Then $f \equiv 0$ in $|z| < R$.*